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On the Zeros of Bessel Polynomials

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A closed expression for the reciprocal power sums of the difference of zeros of Bessel polynomials is derived using elementary complex analysis. These are sums of the form

$$\sum_{\substack{k=1\\k\neq i}}^{n} (x_i - x_k)^{-m}; \qquad m = 1, 2, \ldots$$

where x_i are the zeros of a Bessel polynomial. Recurrence formulae for sums $\sum_{i=1}^{n} x_i^{-m}$ and $\sum_{i=1}^{n} x_i^{m}$ are also established.

1. INTRODUCTION

The differential equation

$$f''(x) + Q(x)\,f'(x) + R(x)\,f(x) = 0 \tag{1.1}$$

has polynomial solutions of exact degree n,

$$f(x) = y_n(x) = \sum_{m=0}^{n} \frac{(n+m)!}{2^m (n-m)!\, m!} x^m \tag{1.2}$$

when $Q(x) = 2(x+1)x^{-2}$ and $R(x) = n(n+1)x^{-2}$. These polynomial solutions seem to have been considered first by S. Bochner [8], who pointed out their connection with Bessel functions. They are also mentioned in a paper by W. Hahn [12]. H. L. Krall and O. Frink [14] were the first to study these polynomials systematically and name them Bessel polynomials.

Also, a polynomial solution of (1.1) of exact degree n is

$$f(x) = \theta_n(x) = \sum_{m=0}^{n} \frac{(n+m)!}{2^m (n-m)!\, m!} x^{n-m}, \tag{1.3}$$

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when $Q(x) = -2(x+n)x^{-1}$ and $R(x) = 2nx^{-1}$. The polynomial $\theta_n(x)$ is related to $y_n(x)$ by

$$\theta_n(x) = x^n y_n(1/x) \tag{1.4}$$

and hence the two are inverse to each other. The polynomials $\theta_n(x)$ were first studied by Burchnall and Chaundy [10] and the relation (1.4) was established by Burchnall [9]. They are also related to the modified Bessel function $K_{n+1/2}(x)$ by

$$\theta_n(x) = \sqrt{2/\pi}\, e^x x^{n+1/2} K_{n+1/2}(x). \tag{1.5}$$

In this paper we shall present nonlinear algebraic equations of the type

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} (x_k - x_j)^{-m} = \phi_m(x_j), \qquad j = 1, 2, \ldots, n, \tag{1.6}$$

satisfied by the zeros of $y_n(x)$ and $\theta_n(x)$.

2. Nonlinear Algebraic Equations

Recently several techniques have been developed to obtain algebraic equations of the type (1.6) for the zeros of classical orthogonal polynomials [2, 7, 15], Bessel functions [3, 7], and confluent hypergeometric functions [4, 7]. In the complex integration technique we start by considering the meromorphic function of the complex variable x,

$$F(x) = (x - x_j)^{-m} f'(x)/f(x) \tag{2.1}$$

which, of course, has $n-1$ simple poles at $x = x_k$, $k = 1, 2, \ldots, j-1, j+1, \ldots, n$ and a pole of order $m+1$ at $x = x_j$. The integral of $F(x)$ over a circle of radius R tends to zero as $R \to \infty$ since $F(x) = O(|x|^{-m-1})$ as $|x| \to \infty$. Consequently, by classical Complex Analysis, the sum of all its residues should vanish. Hence

$$\lim_{R \to \infty} \left[\frac{1}{2\pi i} \int_{Re^{i\theta}} F(z)\, dz \right] = r_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} r_k = 0 \tag{2.2}$$

where r_p is the residue of F at $x = x_p$, $p = 1, 2, \ldots, n$. The residue at each simple pole x_k is given by

$$r_k = \lim_{x \to x_k} [(x - x_k) F(x)]. \tag{2.3}$$

By noting that the polynomial $f(x)$ of degree n can be represented as

$$f(x) = X \sum_{j=1}^{n} (x - x_j) \tag{2.4}$$

and using (2.1) and (2.3), we obtain

$$r_k = (x_k - x_j)^{-m}, \qquad j \neq k. \tag{2.5}$$

On the other hand, at $x = x_j$, there is a pole of order $m+1$ and hence

$$r_j = \frac{1}{m!} \left[\frac{d^m}{dx^m} (x - x_j) \frac{f'(x)}{f(x)} \right]_{x = x_j}; \tag{2.6}$$

(2.2) then implies

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} (x_k - x_j)^{-m} = -\frac{1}{m!} \left[\frac{d^m}{dx^m} (x - x_j) \frac{f'(x)}{f(x)} \right]_{x = x_j} \tag{2.7}$$

which is the general form of nonlinear equation for the zeros x_j of the Bessel polynomials. For instance, for $m = 1$, this yields

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} (x_k - x_j)^{-1} = -\frac{1}{2} \frac{f''(x_j)}{f'(x_j)} = \frac{1}{2} Q(x_j) \tag{2.8}$$

where we have used

$$\frac{1}{R(x_j)} f''(x_j) + \frac{Q(x_j)}{R(x_j)} f'(x_j) = 0 \tag{2.9}$$

implied by (1.1) since $f(x_j) = 0$. Denoting the zeros of $y_n(x)$ by y_j and of $\theta_n(x)$ by z_j, we obtain the first set of nonlinear algebraic equations,

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} (y_k - y_j)^{-1} = y_j^{-1} + y_j^{-2}, \qquad j = 1, 2, \ldots, n; \tag{2.10a}$$

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} (z_k - z_j)^{-1} = -1 - n z_j^{-1}, \qquad j = 1, 2, \ldots, n; \tag{2.10b}$$

using, of course,

$$Q(y_j) = 2(y_j + 1) y_j^{-2}; \qquad Q(z_j) = -2(y_j + n) y_j^{-1}. \tag{2.11}$$

For $m=2$, we similarly obtain

$$3\sum_{\substack{k=1\\k\neq j}}^{n}(x_k-x_j)^{-2}=Q'(x_j)+R(x_j)-\frac{1}{4}Q^2(x_j). \tag{2.12}$$

Using

$$R(y_j)=-n(n+1)\,y_j^{-2};\qquad R(z_j)=2nz_j^{-1} \tag{2.13}$$

and (2.11) yields for the zeros y_j and z_j of $y_n(x)$ and $\theta_n(x)$, respectively, the equations

$$3\sum_{\substack{k=1\\k\neq j}}^{n}(y_k-y_j)^{-2}=-[\{n(n+1)+3\}\,y_j^2+6y_j+1]\,y_j^{-4}, \tag{2.14a}$$

$$3\sum_{\substack{k=1\\k\neq j}}^{n}(z_k-z_j)^{-2}=-1-n(n-2)\,z_j^{-2},\qquad j=1,2,\ldots,n. \tag{2.14b}$$

For $m=3$ we similarly obtain

$$8\sum_{\substack{k=1\\k\neq j}}^{n}(x_k-x_j)^{-3}=Q''(x_j)+2R'(x_j)-Q'(x_j)\,Q(x_j) \tag{2.15}$$

or, equivalently,

$$2\sum_{\substack{k=1\\k\neq j}}^{n}(y_k-y_j)^{-3}=[\{n(n+1)+2\}\,y_j^2+6y_j+2]\,y_j^{-5}, \tag{2.16a}$$

$$8\sum_{\substack{k=1\\k\neq j}}^{n}(z_k-z_j)^{-3}=4n(n-1)\,z_j^{-3}, \tag{2.16b}$$

for the zeros y_j and z_j of the Bessel polynomials $y_n(x)$ and $\theta_n(x)$, respectively.

3. Reciprocal Power Sums

Formulae for reciprocal power sums for the zeros of Bessel polynomials

$$\sigma_j^{(p)}=\sum_{j=1}^{n}x_j^{-p} \tag{3.1}$$

are derived by establishing a recurrence relation for these sums using the first set of nonlinear equations (2.8) and some algebraic manipulations. This technique is essentially a generalization of earlier methods used to derive reciprocal power sums for the zeros of classical orthogonal polynomials [1, 5], Bessel functions [6], and confluent hypergeometric functions [4]. Indeed, multiplying (2.8) by x_j^{-p} and summing over j we get

$$\sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-p}(x_k-x_j)^{-1} = \frac{1}{2} \sum_{j=1}^{n} x_j^{-p} Q(x_j). \tag{3.2}$$

The l.h.s. of (3.2) can be manipulated further using the identity

$$(ab)^{-1} = (a^{-1}+b^{-1})(a+b)^{-1} \tag{3.3}$$

to obtain

$$\begin{aligned}\sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-p}(x_k-x_j)^{-1} &= \sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-(p-1)}(x_j^{-1}+(x_k-x_j)^{-1})x_k^{-1}\\ &= \sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-p}x_k^{-1} + \sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-(p-1)}x_k^{-1}(x_k-x_j)^{-1}.\end{aligned} \tag{3.4}$$

If the identity (3.3) is repeatedly used, we obtain finally

$$\begin{aligned}\sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-p}(x_k-x_j)^{-1} &= \sum_{s=0}^{r-1} \sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-(p-s)}x_k^{-(s+1)}\\ &\quad + \sum_{j=1}^{n} \sum_{k=1}^{n} x_j^{-(p-r)}x_k^{-r}(x_k-x_j)^{-1}.\end{aligned} \tag{3.5}$$

Adding and subtracting the term with $k=j$ in the first triple sum on the right side and using (2.8) gives

$$\begin{aligned} r\sum_{j=1}^{n} x_j^{-(p+1)} &= \sum_{s=0}^{r-1}\left(\sum_{j=1}^{n} x_j^{-(p-s)}\right)\left(\sum_{j=1}^{n} x_j^{-(s+1)}\right)\\ &\quad + \sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-(p-r)}x_k^{-r}(x_k-x_j)^{-1} - \frac{1}{2}\sum_{j=1}^{n} x_j^{-p}Q(x_j).\end{aligned} \tag{3.6}$$

For even p $(=2r)$ the expression (3.6) simplifies to

$$p\sigma_j^{(p+1)} = 2\sum_{s=0}^{p/2} \sigma_j^{(p-s)}\sigma_j^{(s+1)} - \sum_{j=1}^{n} x_j^{-p}Q(x_j) - 2\sigma_j^{(p/2)}\sigma_j^{(p/2+1)}. \tag{3.7}$$

This follows by noting that the middle double sum on the right side of (3.6) vanishes by antisymmetry and by adding and subtracting there a term with $\Delta = r$.

For odd p $(=2r+1)$ the double sum on the right side of (3.6) reduces after some algebraic manipulation to

$$\begin{aligned}\sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} x_j^{-(p+1)/2} x_k^{-(p-1)/2} (x_k - x_j)^{-1} &= \frac{1}{2} \sum_{j=1}^{n} \sum_{\substack{k=1\\ k\neq j}}^{n} (x_j x_k)^{-(p+1)/2} \\ &= \frac{1}{2} (\sigma_j^{((p+1)/2)})^2 - \frac{1}{2} \sigma_j^{(p+1)}\end{aligned} \tag{3.8}$$

and we obtain for odd p

$$p\sigma_j^{(p+1)} = 2 \sum_{s=0}^{(p-1)/2} \sigma_j^{(p-s)} \sigma_j^{(s+1)} - (\sigma_j^{((p+1)/2)})^2 - \sum_{j=1}^{n} x_j^{-p} Q(x_j). \tag{3.9}$$

Equations (3.7) and (3.9) can be combined to give the general recurrence relation

$$p\sigma_j^{(p+1)} = 2 \sum_{s=0}^{[p/2]} \sigma_j^{(p-s)} \sigma_j^{(s+1)} - \sum_{j=1}^{n} x_j^{-p} Q(x_j) - \Sigma_j^{(p)} \tag{3.10}$$

where

$$\begin{aligned}\Sigma_j^{(p)} &= 2\sigma_j^{(p/2)} \sigma_j^{(p/2+1)}, \qquad & p \text{ even}, \\ &= [\sigma_j^{((p+1)/2)}]^2, \qquad & p \text{ odd},\end{aligned} \tag{3.11}$$

and [ ] denotes integral part.

For the zeros of Bessel polynomials, the recurrence relations for the reciprocal power sums follow easily from (3.10), using (2.11):

(i) For z_j:

$$(p - 2n)\, \sigma_j^{(p+1)} = 2\sigma_j^{(p)} + 2 \sum_{s=0}^{[p/2]} \sigma_j^{(p-s)} \sigma_j^{(s+1)} - \Sigma_j^{(p)}. \tag{3.12}$$

(ii) For y_j:

$$2\sigma_j^{(p+2)} = -(p+2)\, \sigma_j^{(p+1)} + 2 \sum_{s=0}^{[p/2]} \sigma_j^{(p-s)} \sigma_j^{(s+1)} - \Sigma_j^{(p)}. \tag{3.13}$$

Thus for the zeros z_j of $\theta_n(x)$ we obtain the earlier results [11]

$$\sigma_j^{(1)} = -1; \qquad \sigma_j^{(2)} = \frac{1}{2n-1}; \qquad \sigma_j^{(3)} = 0;$$
$$\sigma_j^{(4)} = \frac{-1}{(2n-1)^2(2n-3)}; \qquad \sigma_j^{(5)} = 0; \qquad \text{etc.} \tag{3.14}$$

In the case of $y_n(x)$, we first determine $\sigma_j^{(1)}$ using Eq. (2.10a) so that the higher-order sums are derivable from the recurrence formula (3.13). Indeed, multiplying (2.10a) by y_j and summing over j, we obtain

$$\sum_{j=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} y_j(y_k - y_j)^{-1} = n + \sum_{j=1}^{n} y_j^{-1}. \tag{3.15}$$

Noting that the left-hand side equals $-n(n-1)/2$, we get

$$\sigma_j^{(1)} = \sum_{j=1}^{n} y_j^{-1} = -n(n+1)/2. \tag{3.16}$$

The recurrence formula (3.13) then gives

$$\sigma_j^{(2)} = n(n+1)/2; \qquad \sigma_j^{(3)} = \tfrac{1}{8}n(n+1)(n+3)(n-2);$$
$$\sigma_j^{(4)} = -\tfrac{1}{2}n(n+1)(n^2+n-3); \qquad \text{etc.} \tag{3.17}$$

Concerning the zeros of $\theta_n(x)$ we can immediately conclude that the odd-powered sums vanish [9, 11]:

$$\sigma_j^{(l)} = 0 \qquad \text{for} \quad l = 3, 5, \ldots, 2n-1. \tag{3.18}$$

This may be observed by considering (3.12) for even p which may also be written as

$$(p-2n)\,\sigma_j^{(p+1)} = 2 \sum_{s=1}^{(p-2)/2} \sigma_j^{(p-s)} \sigma_j^{(s+1)}. \tag{3.19}$$

Thus, for $p=4$, $\sigma_j^{(5)}$ is expressed in terms of $\sigma_j^{(3)}$ which vanishes, and hence $\sigma_j^{(5)} = 0$. Continuing in this fashion we notice that the right-hand side of (3.19) equals zero for $p = 4, 6, 8, \ldots, 2n-2$ and (3.18) follows. For $p = 2n$ we, however, obtain

$$\sum_{s=1}^{n-1} \sigma_j^{(2n-s)} \sigma_j^{(s+1)} = 0 \tag{3.20}$$

which, of course, confirms (3.18). But, for $p=2n+2$,

$$\sigma_j^{(2n+3)}=\sum_{s=1}^{n}\sigma_j^{(2n+2-s)}\sigma_j^{(s+1)}=\sigma_j^{(2n+1)}\sigma_j^{(2)}$$
$$+\sum_{s=2}^{n}\sigma_j^{(2n+2-s)}\sigma_j^{(s+1)}. \tag{3.21}$$

The second term on the right-hand side of (3.21) involves odd-powered sums which vanish and we obtain the result of Ismail and Kelker [13]

$$\sigma_j^{(2n+3)}=\frac{1}{2n-1}\sigma_j^{(2n+1)}.$$

In this way many other results connecting the reciprocal power sums of the zeros can be derived from the recurrence formulae (3.12) and (3.13).

4. Other Sum Rules

Since the Bessel polynomials $\theta_n(x)$ and $y_n(x)$ are inverse to each other, the reciprocal power sums for the zeros of $\theta_n(x)$ are the power sums for the zeros of $y_n(x)$ and vice versa. Hence (3.12) is the recurrence formula for the power sums of the zeros y_j of $y_n(x)$, where now

$$\sigma_j^{(p)}=\sum_{j=1}^{n}y_j^p, \qquad p=1,2,\ldots. \tag{4.1}$$

Thus, for instance,

$$\sum_{j=1}^{n}y_j=-1; \qquad \sum_{j=1}^{n}y_j^2=\frac{1}{2n-1}; \qquad \sum_{j=1}^{n}y_j^3=0. \tag{4.2}$$

Similarly the recurrence formula for the power sums of the zeros z_j of $\theta_n(x)$ is given by (3.13) where now

$$\sigma_j^{(p)}=\sum_{j=1}^{n}z_j^p. \tag{4.3}$$

Another interesting result for the zeros y_j of $y_n(x)$ follows from (2.14a) and (3.17), namely,

$$4\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}(y_j-y_k)^{-2}=-n(n+1)(n^2+n-2). \tag{4.4}$$

For the zeros z_j of $\theta_n(x)$ we similarly obtain

$$3 \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} (z_j - z_k)^{-2} = -n(n-1)(2n-3). \tag{4.5}$$

The odd-powered double sum

$$\sum_{j=1}^{n} \sum_{\substack{l=1 \\ l \neq j}}^{n} (x_j - x_l)^{-(2k+1)}, \qquad k = 1, 2, \ldots, \tag{4.6}$$

of course vanishes by antisymmetry as can also be proved using our previous results.

Conclusions

We have reported nonlinear algebraic equations of the form

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} (x_j - x_k)^{-m} = \phi_m(x_j)$$

for $m = 1, 2, 3$ for the zeros x_j of the Bessel polynomials. A general technique to derive these sums using a determinant property is presented in [7] for all m. From (1.5) it is clear that the Bessel polynomials $\theta_n(x)$ and the modified Bessel functions $K_\nu(x)$ have common zeros whenever $\nu = n + \frac{1}{2}$, n being a nonnegative integer. Hence the results presented here for the zeros of Bessel polynomials can be translated immediately into results for the zeros of $K_{n+1/2}(x)$ which are also known as spherical Bessel functions. The recurrence formula (3.10) for the reciprocal power sums (3.1) derived here for the zeros of Bessel polynomials can also be used to obtain such sums for other polynomials satisfying the differential equation (1.1). These, of course, include the classical orthogonal polynomials.

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